THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MATH3070 Introduction to Topology 2017-2018
Solution of Tutorial Classwork 2

1. Note that

$$\begin{aligned} U \in \mathfrak{T}_X|_A \times \mathfrak{T}_Y|_B &\iff U = \cup_{\alpha \in I}(V_\alpha \times W_\alpha) \text{ for some } V_\alpha \in \mathfrak{T}_X|_A, W_\alpha \in \mathfrak{T}_Y|_B \\ &\iff U = \cup_{\alpha \in I}((\tilde{V}_\alpha \cap A) \times (\tilde{W}_\alpha \cap B)) \text{ for some } \tilde{V}_\alpha \in \mathfrak{T}_X, \tilde{W}_\alpha \in \mathfrak{T}_Y \\ &\iff U = \cup_{\alpha \in I}((\tilde{V}_\alpha \times \tilde{W}_\alpha) \cap (A \times B)) \text{ for some } \tilde{V}_\alpha \in \mathfrak{T}_X, \tilde{W}_\alpha \in \mathfrak{T}_Y \\ &\iff U \in \mathfrak{T}_{X \times Y}|_{A \times B} \end{aligned}$$

Hence $\mathfrak{T}_X|_A \times \mathfrak{T}_Y|_B = \mathfrak{T}_{X \times Y}|_{A \times B}$.

Remark: Note that in the tutorial classwork delivered in the class, the definition of product topology is given by

$$\mathfrak{T}_{X \times Y} = \{U \times V \mid U \in \mathfrak{T}_X, V \in \mathfrak{T}_Y\}, \qquad (*)$$

This definition is **INCORRECT**. In fact it is not a topology. To show a counter example, take $(X, \mathfrak{T}_X) = (Y, \mathfrak{T}_Y) = (\mathbb{R}, \mathfrak{T}_{\text{std}})$. Under the definition of (*), clearly we have

$$A = (0, 1) \times (0, 1) \in \mathfrak{T}_{X \times Y} \text{ and } B = (2, 3) \times (0, 2) \in \mathfrak{T}_{X \times Y}.$$

If (*) is a topology, we will have $A \cup B \in \mathfrak{T}_{X \times Y}$. However, one can show that $A \cup B \neq U \times V$ for any $U, V \in \mathfrak{T}_{\text{std}}$. So $A \cup B \notin \mathfrak{T}_{X \times Y}$, contradiction.

2. Pick any $x \in \overline{A}$ in $(X, \mathfrak{T}_X)$. Then for any open set $U \in \mathfrak{T}_X$ with $x \in U$, we have $U \cap A \neq \emptyset$. Since $A \subset Y$ and Y is closed in X, we have $\overline{A} \subset Y$. In particular, this implies $(U \cap Y) \cap A = U \cap A \neq \emptyset$. Therefore, for any $W \in \mathfrak{T}_Y$ with $x \in W$, we have $W \cap A \neq \emptyset$. Hence, $x \in \overline{A}$ in $\mathfrak{T}_Y$. Since A is closed in Y, we have $x \in \overline{A} = A$. Altogether, we have $\overline{A} \subset A$ and hence $A = \overline{A}$.

3. (a) Pick any $y \in Y$. Let $V \in \mathfrak{T}_Y$ with $y \in V$. Pick any $x \in f^{-1}(y)$. By continuity of f, we have $x \in f^{-1}(y) \subset f^{-1}(V) \in \mathfrak{T}_X$. Since $x \in X = \overline{D}$, we have $f^{-1}(V) \cap D \neq \emptyset$. Hence $f(f^{-1}(V) \cap D) = V \cap f(D) \neq \emptyset$. Therefore, we have $y \in \overline{f(D)}$. Since y is arbitrary, $f(D)$ is a dense subset in Y.

   (b) No. For example, take $(X, \mathfrak{T}_X) = (\mathbb{R}, \mathfrak{P}(X))$, $(Y, \mathfrak{T}_Y) = (\mathbb{R}, \mathfrak{T}_{\text{std}})$ and $f = id_{\mathbb{R}}$. Since $\mathfrak{T}_X$ is the discrete topology, f is automatically continuous. For the dense subset $\mathbb{Q} \subset Y$, we have $f^{-1}(\mathbb{Q}) = \mathbb{Q}$ in X. However, since $\sqrt{2} \in \{\sqrt{2}\} \in \mathfrak{T}_X$ and $\{\sqrt{2}\} \cap \mathbb{Q} = \emptyset$, $\mathbb{Q}$ is not dense in X.

   Remark: Similarly, you can find infinity many counter examples by considering $\mathbb{R}^n$ and $\mathbb{Q}^n$.